Development of steady flow of liquid film over a domed cylinder

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Development of steady flow of a uniform liquid film over a domed cylinder in both laminar and turbulent regimes was formulated and solved. Depending on the inlet condition, the flow may start with the accelerating or decelerating mode and arrive at the terminal velocity asymptotically.

Keywords: liquid film flow; thin film flow; development of thin film flow

Introduction

A passive containment cooling system for a nuclear reactor (EPRI 1989) consists of flow of a thin water film over a cylinder topped by a dome formed by an ellipsoid of revolution at a mean velocity of less than 1 m/s. The film thickness (of the order of 1 mm or less) is much smaller than the radius of the cylinder (of the order of 10 m). Therefore, flow over the body of revolution can be treated as that over an inclined plane. Uniform wetting and flow were assured by a surface coating and at a flow rate from the top of the dome over the entire vessel given by the criterion of Hartley and Murgatroyd (1964) based on surface interactions. This note treats the basic interactions in the flow system and gives solutions to the cases of symmetric film flow. It provides an understanding of the problem to facilitate the efforts of numerical computation.

The geometry of a containment vessel can be represented by the domed cylinder as shown in Figure 1, giving the coordinate system. The balance of averaged momentum in a developing symmetric liquid film flow over this containment vessel is approximated by

$$u_m du_m/dx = -(\tau/\rho b) + g \sin\theta \qquad (1)$$

neglecting the friction of the air outside the film (nearly 4 percent of that at the wall in a practical system) and the effect of surface tension or waves (Fulford 1964; Faghri and Payvar 1979) and the effect of heat and mass transfer, for the sake of simplicity in discussing the solution in closed forms. Moreover, there is no pressure drop along the x-direction because of the lack of a free stream of the liquid (Levich 1964). In addition, u_m is the mean film velocity, x is the coordinate along a meridian, ρ is the density of the liquid, b is the thickness of the film, g is the gravitational acceleration, θ is the angle made by the flow direction with that normal to the gravity, and τ is the shear resistance at the wall due to viscosity of the liquid (Fulford 1964). When the film reaches its terminal velocity, we get

$$\tau = g\rho b \sin\theta = f\rho u_m^2/2 \qquad (2)$$

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where f is the friction factor. We further note that when the fully developed velocity of the film is reached, the shear stress supports the body force per unit area (Fulford 1964), or

$$f = gb \sin\theta/(u_m^2/2) \qquad (3)$$

The film thickness b and the volumetric flow rate per unit width Q_L are related according to:

$$bu_m = Q_L = Q/2\pi r \qquad (4)$$

where Q is the total flow rate, r is the radius, which is a variable in the domed section, and radius $r = R$ of the cylindrical section.

The above relations are applicable to both laminar and turbulent flow regimes. The only assumption is that with the velocity distribution in the film flow, the rate of change of momentum is given by the average velocity in Equation 1. The degree of approximation due to such an assumption will be

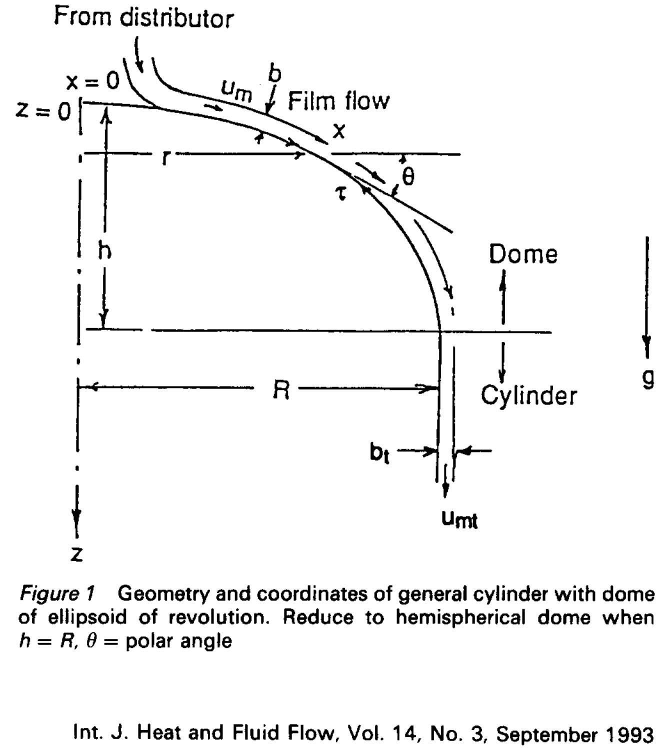

Figure 1 Geometry and coordinates of general cylinder with dome of ellipsoid of revolution. Reduce to hemispherical dome when $h = R$, θ = polar angle

discussed later. The friction factors, however, differ according to the flow regimes.

For laminar flow, neglecting the friction of the air outside of the film, the velocity distribution in the film is parabolic, being zero at the wall. The inlet length for the development of the boundary layer inside the film has been shown to be extremely small (Appendix 1). For flow down an inclined plane, Nusselt (see Fulford 1964) gave the mean terminal velocity:

$$u_{mt} = g \sin \theta b^2/3\nu \tag{5}$$

where ν is the kinematic viscosity. Combining Equation 5 with Equation 4 gives

$$u_{mt} = [(g \sin \theta/3\nu)Q_L^2]^{1/3} \tag{6}$$

and the terminal film thickness

$$b_t = (Q_L 3\nu/g \sin \theta)^{1/3} \tag{7}$$

These relations give, for Reynolds number Re given by Q_L/ν,

$$f_L = g \sin \theta b_t/(u_{mt}^2/2) = 6\nu/Q_L = 6/\mathrm{Re} \tag{8}$$

Laminar film flow exists over most of the operating range of a containment cooling system; turbulent flow may occur at the upper range of flow and at elevated temperatures, which leads to reduced kinematic viscosity. For turbulent flow, in spite of the facts that the turbulence in the film is dampened by the liquid surface tension and the velocity profile of the film is not well known, one gets, from the turbulent velocity profile given in Schlichting (1979),

$$\tau_T = (0.0225)\ 2\ (\rho u_s^2/2)(\nu/u_s y)^{1/4} = f_T(1/2)\rho u_m^2 \tag{9}$$

where u_s is now the velocity at the liquid surface, $u_m \sim 0.817u_s$ for a 1/7th velocity profile, and y is nearly the displacement thickness and is equal to $b/3.9$ for the present system. Equation 9 gives

$$f_T = g \sin \theta b_t/(u_{mt}^2/2) = 0.09/\mathrm{Re}^{1/4} \tag{10}$$

This confirms the transition Reynolds number of 270 given in Fulford (1964): both Equations 8 and 10 give $f = 0.0222$ at Re = 270. For turbulent flow we get

$$u_{mt} = [g \sin \theta Q_L^{5/4}/0.045\nu^{1/4}]^{1/3} \tag{11}$$

and

$$b_{mt} = [0.045\nu^{1/4}Q_L^{7/4}/g \sin \theta]^{1/3} \tag{12}$$

Since wave motion in the film is neglected, the solution of Equation 1 gives the average velocity and thickness of the developing liquid film flow.

Symmetric film flow over an ellipsoid of revolution

For minor radius h and major radius R, the coordinates r and z are related by

$$(r/R)^2 + [1 - (z/h)]^2 = 1 \tag{13}$$

which gives, for z less than h,

$$dz = (h/R)^2(h - z)^{-1} r\, dr \tag{14}$$

The meridian coordinate x is given by

$$(dx)^2 = (dr)^2 + (dz)^2 \tag{15}$$

and $dz = dx$ for $z \geq h$; substitution of Equation 14 gives (let $1 - (h/R)^2 = a$)

$$dx = [(1 - a(r/R)^2]^{1/2}[1 - (r/R)^2]^{-1/2}\, dr \tag{16}$$

The angle θ in Figure 2 is given by

$$\tan \theta = dz/dr = (h/R)(r/R)[1 - (r/R)^2]^{-1/2} \tag{17}$$

and $\sin \theta$ is now given by

$$\sin \theta = (h/R)(r/R)[1 - a(r/R)^2]^{-1/2} \tag{18}$$

Notation

b	Film thickness
f	Friction factor
F	Reduced Froude number
Fr	Froude number of film
g	Gravitational acceleration
G	A function defined by Equation 33
h	Height of dome
Q	Volume flow rate
Q_L	Volume flow rate per unit width
r	Radial coordinate or radius
R	Radius of dome and cylinder
R^*	Dimensionless quantities for film flow, laminar (sub L) or turbulent (sub T)
Re	Reynolds number of film
u	Longitudinal component of velocity in the film
v	Normal component of velocity
x, y	Coordinates as defined
X	Dimensionless longitudinal coordinate
z	Axial coordinate or direction of gravity

Greek symbols

δ	Boundary layer thickness inside the film or condensing liquid film thickness
δ_x^*	Displacement thickness of the boundary layer as defined in Equation A.2
θ	Angle made with direction normal to gravity, or polar angle
θ_x	Momentum thicknesses of the boundary layer as defined in Equation A.3
ν	Kinematic viscosity
ρ	Density
τ	Shear stress

Subscripts (properties without subscript refer to water or liquid)

a	Air
L	Laminar
m	Mean
o	Reference quantity or outer passage
R	At radius R
s	Surface of liquid film
t	Terminal
T	Turbulent
w	Wall

Superscript

+	Dimensionless quantities

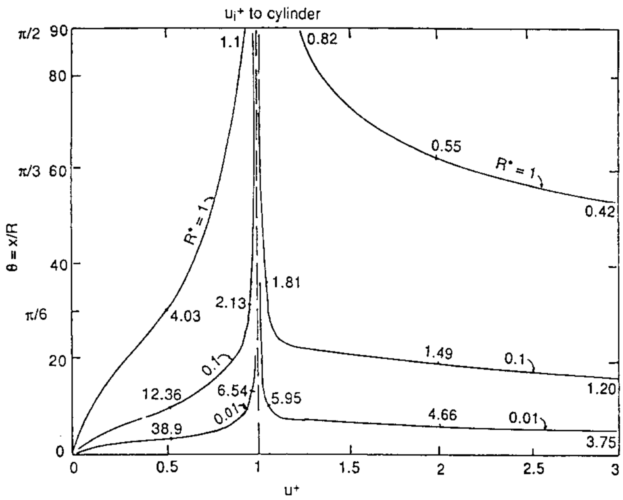

Figure 2 Symmetric flow over a hemispherical dome for laminar or turbulent flow at $R^* = 1, 0.1, 0.01$, based on conservative friction factor ($u^+ = u_m/u_{mtR}$, $b^+ = b/b_{tR} = 1/u^+ \sin\theta$ are marked in numerals)

Note that for a spherical dome, $R = h$, $dx = [1-(r/R)^2]^{-1/2}\,dr$, or $x = R\sin^{-1}(r/R) = R\theta$. θ is the polar angle, or $\sin\theta = r/R$. The cylindrical part begins as θ reaches $\pi/2$.

Equation 1 can be written as

$$u_m\,du_m/dx = -f(u_m^2/2b) + g\sin\theta \tag{19}$$

Introducing $u^+ = u_m/u_{mtR}$ (where u_{mtR} is given by u_{mt} at $r = R$ and $\theta = \pi/2$ for nondimensionalizing) and $r^+ = r/R$, together with Equation 4, Equation 19 can be rearranged as

$$(u_{mtR}^2/g)u^+du^+/dx = \sin\theta - (f/2)(2\pi r/Qg)u_{mtR}^3 u^{+3} \tag{20}$$

Note that f depends on the flow regime as given by Equations 8 and 10. Equation 20 now becomes

$$\begin{aligned} R^*u^+du^+/dr^+ &= (h/R)r^+(1-r^{+2})^{-1/2} \\ &\quad - r^{+n}[1-ar^{+2}]^{1/2}(1-r^{+2})^{-1/2}u^{+3} \end{aligned} \tag{21}$$

where, for laminar flow, $n = 2$, and R^* is given by

$$R_L^* = [(Q/2\pi)^4/9R^7\nu^2 g]^{1/3} = [\mathrm{Re}_R\mathrm{Fr}_R/3]^{2/3} \tag{22}$$

where $\mathrm{Re}_R = Q/2\pi R\nu = Q_{LR}/\nu$ and the Froude number based on characteristic dimension R is defined as $\mathrm{Fr}_R \equiv (Q/2\pi R)/(R^3 g)^{1/2}$; R^* is therefore a parameter correlating the effect of gravity and viscosity for given volumetric flow rate per unit width of film. For turbulent flow, $n = 5/4$, and R^* is given by

$$R_T^* = [(Q/2\pi)^{5/2}/0.045^2 R^{11/2}\nu^{1/2}g]^{1/3} = [\mathrm{Re}_R^{1/4}\mathrm{Fr}_R/0.045]^{2/3} \tag{23}$$

with turbulent flow at $\mathrm{Re} = (Q/2\pi r)/\nu = \mathrm{Re}_R/r^+$ of greater than 270 where $R_T^* = R_L^*$; transition to laminar flow may occur as the film proceeds downward. A comparison of the above approximation based on Equation 1 to that based on momentum integral method is given in Appendix 1, together with the effect of boundary-layer growth.

Integration of Equation 21 is to be carried out for each combination of system parameters R^* and (h/R) for laminar and turbulent flow, from $r^+ = 0$ to 1, for the range $0 \le u^+ \le 1$ for accelerating flow, with a series expansion from $r^+ = 0$, and for the range $1 \le u^+ \le \infty$, or decelerating flow, starting from an asymptotic expansion from large u^+. These are followed by numerical computation to $r^+ = 1$. $r^+ = 1$ is at the edge of the dome, where the transition to flow over the cylinder occurs. Note that for a given dome, when R^* is large (or there is large volumetric liquid flow), the film will reach the rim at a velocity more different from the terminal velocity u_{mt} than for a small R^*. The initial value of u_i^+ is given by the physical inlet at r_i^+ for given Q, R, h, and ν for accelerating flow. In designing a water film cooling system, R^* may have the range of 10^{-3} to 10^{-1}; the velocity reached at the rim is the inlet velocity to the cylindrical section. Instead of numerical results, the solution is illustrated with the case of a hemispherical dome.

Hemispherical dome

A special case of the above system is a hemispherical dome, where $h/R = 1$, and θ is also the polar angle. If we take the conservative estimate of the friction factor based on terminal flow condition over radius R, Equation 21 reduces to

$$R^*u^+du^+/d(x/R) = \sin(x/R) - \sin(x/R)u^{+3} \tag{24}$$

with $n = 1$ for both laminar and turbulent flow. This gives a one-parameter family of curves for various R^* depending on the flow regime. Equation 24 is integrable with $x/R = \theta$; θ_i, u_i^+ at the inlet, up to $\theta = \pi/2$ at the rim. This gives

$$\begin{aligned} &(\cos\theta_i - \cos\theta)/R^* \\ &\quad = 3^{-1/2}\tan^{-1}\{3^{1/2}(u_i^+ - u^+)/[2u^+u_i^+ + (u^+ + u_i^+) + 2]\} \\ &\qquad + 6^{-1}\ln\{[(1-u_i^+)/(1-u^+)]^3[(1-u^{+3})/(1-u_i^{+3})]\} \end{aligned} \tag{25}$$

for computating u^+ from θ_i to $\theta = \pi/2$ at the rim of the dome. For a general discussion, one has for $u_i = 0$ at $\theta_i = 0$ (accelerating flow)

$$\begin{aligned} 1-\cos\theta = R^*\{&-3^{-1/2}\tan^{-1}[3^{-1/2}(2u^+ + 1)] \\ &+ 6^{-1}\ln[(1-u^{+3})/(1-u^+)^3]\} \end{aligned} \tag{26}$$

and for $u_i = \infty$ at $\theta_i = 0$ (decelerating flow)

$$\begin{aligned} 1-\cos\theta = R^*\{&3^{-1/2}(\pi/2) - 3^{-1/2}\tan^{-1}[3^{-1/2}(2u^+ + 1)] \\ &+ 6^{-1}\ln[(1-u^{+3})/(1-u^+)^3]\} \end{aligned} \tag{27}$$

for determining u^+ at θ for any given u_i^+ and θ_i.

Figure 2 gives the general relation for various values of R^*. Where a curve of a given value of R^* meets the line $\theta = \pi/2$ gives the inlet u_i^+ to the cylinder. The film thickness b/b_t is given by $1/u^+\sin\theta$; b_t is the terminal film thickness. The trend of change of film thickness is seen by the numerals marked on the curves of various R^* for u^+, increasing in decelerating flow and thinning in accelerating flow. Since practical values of R^* in the proposed containment (EPRI 1989) is of the order of 10^{-3} to 10^{-2} for a spherical dome, the flow reaches its terminal velocity over a small polar angle, with film thickness adjusted according to Equation 4. One also notes that the approximations in Equations 26 and 27 give a conservative estimate for decelerating flow because the local Reynolds number is higher over the dome than over the cylindrical wall.

The price of averaging as in Equation 1 is that a derivation based on momentum integral method, a more logical averaging procedure than that of Equation 1, gives a coefficient of 9/10 on the left-hand side of Equation 24 for laminar flow (R^* for laminar flow may be modified by multiplying by 9/10); for turbulent flow, this coefficient will be even closer to 1. Another approximation is the use of a maximum friction factor for the system, which makes possible a closed-form solution for flow over the spherical dome.

Straight cylinder

Setting $x/R = \pi/2$ reduces Equation 24 to the case of flow over the vertical cylinder or flat plate ($x = z$):

$$u^+ du^+/dX = 1 - u^{+3} \tag{28}$$

where X is given by

$$X = (x/R)/R^* \tag{29}$$

since $(Q/2\pi R)$ in Equation 22 or 23 reverts to Q_L, R cancels out, and $X = (Q_L^4/9\nu^2 g)^{-1/3}x$ for laminar flow, for instance. Equation 28 is readily integrated in closed form for both flow regimes represented by pertinent relations for R^*:

$$0 \le u^+ \le 1: \quad X = (1/6)\ln[(1 - u^{+3})/(1 - u^+)^3] - 3^{-1/2}\tan^{-1}[3^{1/2}u^+/(u^+ + 2)] \tag{30}$$

$$1 \le u^+ < \infty: \quad X = (1/6)\ln[(1 - u^{+3})/(1 - u^+)^3] - 3^{-1/2}\tan^{-1}[3^{-1/2}(2u^+ + 1)] + (\pi/3^{1/2}2) \tag{31}$$

where Equation 30 is for accelerating flow and Equation 31 is for decelerating flow. These relations are plotted as shown in Figure 3. A solution to a similar equation was given by Kasimov and Zigmund (Fulford 1964) expressed in terms of b instead of u^+ (note that $b = Q_L/u^+u_{mt}$) for laminar flow only, with determination of X from $X = 0$ at u_i^+. The length for development of terminal flow is now given by a single set of solution of Equation 28:

$$X_{1\%} - X_i = X(u^+_{1\%}) - X(u_i^+) \tag{32}$$

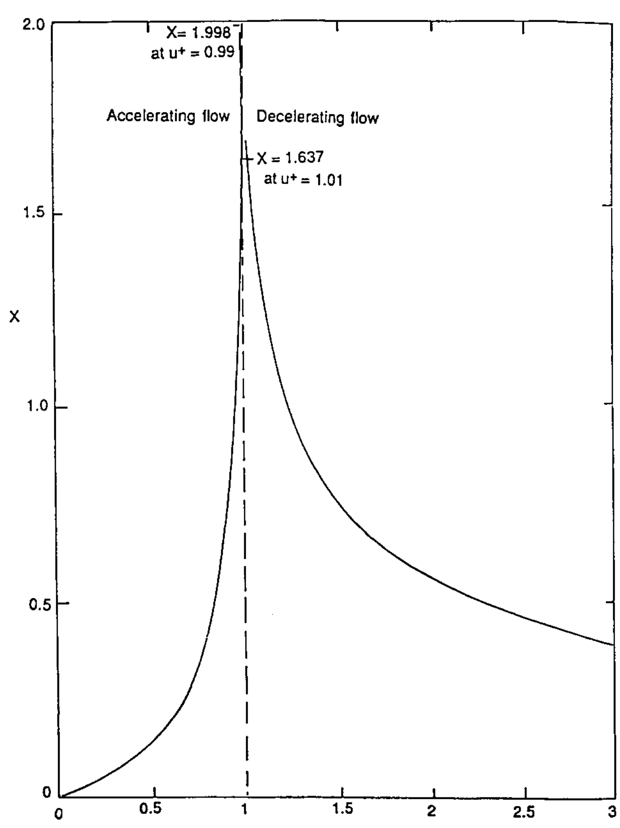

Figure 3 Developing flow over a flat plate or a large cylinder—laminar or turbulent flow, $u^+ = u_m/u_{mt}$

where $X_{1\%}$ and $u^+_{1\%}$ are the values of X at $u^+ = 0.99$ for accelerating flow and at $u^+ = 1.01$ for decelerating flow when the flow is practically at its terminal velocity.

Solution in terms of Froude number of film flow

An alternate method for determining the motion of the film down a domed cylinder is by replacing the equation of motion in terms of the Froude number of the film, $\mathrm{Fr} = u_m/(bg\sin\theta)^{1/2}$. We introduce a terminal Froude number, $\mathrm{Fr}_{tR} = u_{mtR}/(b_{tR}g)^{1/2}$ when the film reaches its terminal velocity some distance down the side of the cylinder. If we express the equation of motion in terms of a dimensionless quantity $F = \mathrm{Fr}/\mathrm{Fr}_{tR}$, the equation of motion over a general ellipsoid of revolution, after substitution of

$$F = G^{3/2}(1 - ar^{+2})^{1/4} \tag{33}$$

takes the form

$$(R^*)(h/r)^{2/3}G\,dG/dr^+ = (h/R)r^+(1 - r^{+2})^{-1/2} - r^{+n}(1 - ar^{+2})^{1/2}(1 - r^{+2})^{-1/2}G^3 \tag{34}$$

where $a = 1 - (h/R)^2$; for the given dome shape (h/R), R^* is as given before. The solution is given by integration from $r^+ = r_i^+$ at the inlet to $r^+ = 1$ at the rim of the dome.

In the case of a *hemispherical dome*, Equation 34 reduces according to $R = h$, $dx^+ = dr^+/(1 - r^{+2})^{1/2}$, and $x = R\sin^{-1}r^+ = R\,\theta$, with θ the polar angle. This gives, with $x^+ = x/R$,

$$R^*G\,dG/d\theta = \sin\theta - \sin\theta\,G^3 \tag{35}$$

with $0 < x^+ < \pi/2$, with similar simplification as that leading to Equation 24. Note that Equation 35 is integrable in closed form as in Equations 24 and 25. The solution is identical to replacing u^+ by G in Figure 2.

Continuing the above to the *cylindrical portion* beginning at the rim of the dome, Equation 34 further reduces to

$$G\,dG/dX = 1 - G^3 \tag{36}$$

where $X = x^+/R^*$. Integration of Equation 36 starts from G $(r^+ = 1)$ at the rim of the dome, which becomes G_i at X_i of the cylinder with integration to X at a value of G (or F) that is 1 percent away from $G = 1$.

Note that Equation 36 is integrable in closed form, G is equal to $F^{2/3}$, and Equation 36 reduces to the same form as Equation 28. The solution gives, from $F = F_i$ to $F = 1 \pm 0.01$ ("+" for deceleration and "−" for acceleration),

$$X = (3^{-1/2})\tan^{-1}\{3^{1/2}(F_i^{2/3} - F^{2/3})/[2F^{2/3}F_i^{2/3} + (F^{2/3} + F_i^{2/3}) + 2]\} + 6^{-1}\ln\{[(1 - F_i^{2/3})/(1 - F^{2/3})]^3[(1 - F^2)/(1 - F_i^2)]\} \tag{37}$$

Only the motion over the dome needs be integrated numerically. This functional relation can also be expressed in the same form as in Equations 30 and 31 for general integration from F_i at X_i to $F_{1\%}$ at $X_{1\%}$. It is then identical to replacing u^+ by $F^{2/3}$ in Figure 3. While the advantage of computing with Fr as a dependent variable is not obvious, formulation by redefining Fr with u_m normal to g may afford an exploration of a possible hydraulic jump over the portion of the dome where θ is small, depending on the inlet condition (Rahman et al. 1991).

Discussion

The effect of the air stream over the liquid film is readily accounted for. As an illustration, the right-hand side of Equation 28 will have an additional term $-(f_a/2)[\rho_a u_a^2/\rho(3\nu g^2 Q/2\pi R)^{1/3}]u^+$, where the subscript a denotes those properties of air. One would then have to be satisfied with a numerical solution.

When the effect of waves is significant, the above gives the average film velocity and thickness of symmetric flow distribution over the domed cylinder.

Effective wetting of the vessel is assisted by the use of a suitable coating and by a large initial flow rate of water. Cutting back to the desired flow for the period of time as designed provides for a stable liquid film less than 1 mm thick given consideration of interactions of surface tension, viscosity, and gravity. It was shown that at a contact angle of 20°, a stable film of water at 300 K is attainable at a minimum flow of 4×10^{-5} m^3/m,s based on a force criterion and a minimum flow of 10^{-4} m^3/m,s according to the energy criterion of Hartley and Murgatroyd (1964). Ideally, a circumferential uniform film with fine wave crests is expected to flow down the wall of the vessel, but its uniformity might be affected adversely by the counterflow of the air in a natural draft and by the nonuniformity of its circumferential distribution (EPRI 1989). A chosen tolerance in the manufacture and coating process may also lead to this hydrodynamic instability due to the slower flow of a thin film than a thick film, which may lead to a flow in streaks over a given height. This condition can be improved by introducing a tangential component of water flow from its distributor. Since this tangential component tends to be dissipated by the wall friction over a relatively short height, it helps to even out initial distribution of water in the film, but will not ensure a uniform film, say, one containment diameter away from the top. Instability of film flow and the large contact angle of liquid with the solid surface may lead to dry patches and nonsymmetric flow distribution.

Thermal instability (Fugita and Ueda 1978) of the liquid film may arise due to nonuniform internal heating in the vessel or nonuniform water film thickness due to an allowed tolerance in the manufacture of the vessel and its surface finish or due to the asymmetry of the natural-draft air flow over the liquid film. It is readily shown that an evaporating liquid film is inherently unstable because a thin spot tends to attain a higher surface temperature than the rest of the film for the same heat flux, leading to a greater evaporation rate that increases as dry-out is approached. This is enhanced by the increase in vapor pressure due to surface tension as the film thins out, in addition to the effect of the surface tension gradient, and may lead to dry patches over the vessel and nonuniform coverage of the liquid film. Conversely, a condensing film tends to remain stable.

Conclusions

Equation 21 for a given geometry of the ellipsoidal dome over its height h or from $r = r_i$ (where $u_m = u_{mi}$) to $r = R$ accounts for the developing flow in terms of u_m over the dome to its edge, depending on the values of R^*. R^* correlates the flow quantity, fluid properties, characteristic dimension, and flow regime. The solution continues into the cylindrical portion of the vessel, where u_m from the dome becomes u_{mi} until the flow develops to practically u_{mt} because of the waves and other disturbances. The closed-form solutions in the above provide a convenient means for computer code check. To account for evaporation of the film, computation can be carried out from given initial conditions with sectionally varying values of R^*.

It is seen that for R^* of the order of 0.01 to 0.001, the liquid film develops to its terminal velocity and thickness over a very short distance. For $R^* < 0.001$, which is the case in most applications, the film flow is practically at its local terminal velocity.

Because of decreasing film Reynolds number as the flow proceeds downward, the transition of the flow regime tends to be from an initial turbulent range to the laminar range as the film Reynolds number decreases. Physically, this may occur at the larger radius as the volumetric flow rate per unit width decreases, or as this rate decreases by evaporation.

An alternate method expresses the equation of motion of the film in terms of its variation in Froude number. In this case, u_m is replaced by $(\mathrm{Fr}/\mathrm{Fr}_t)^{2/3}(h/R)^{1/3}(1 - ar^{+2})^{-1/6}$. However, no advantage in doing this is indicated. A possible hydraulic jump over a shallow dome has to be treated separately.

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Appendix 1. Development of film flow based on approximation based on momentum integral method

A solution based on the momentum integral method (Schlichting, 1979) has been developed for comparison to the above solutions. In the same coordinate system in Figure 1, the equation of momentum integral of the film (neglecting the friction of air on the liquid surface) is given by

$$(\partial/\partial x)(ru_s^2\theta_x) + ru_s(\partial u_s/\partial x)\delta_x^* - r\delta u_s(\partial u_s/\partial x) = r(\tau_{wxy}/\rho) - rg\delta \sin\theta \tag{A.1}$$

It differs from the boundary-layer equation in Schlichting because there is no free stream of the liquid, and $dP/dx = 0$ (Levich 1962). In Equation (A.1), u_s is the surface velocity of the film for the case of laminar boundary-layer motion, $u_s = 3u_m/2$, and the displacement thickness with boundary-layer thickness δ inside b is given by

$$\delta_x^* = \int_0^b [1 - (u/u_s)]dy = (1/3)\delta \tag{A.2}$$

and correspondingly

$$\theta_x = \int_0^b (u/u_s)[1 - (u/u_s)]dy = (2/15)\delta \tag{A.3}$$

until $\delta = b$. We note that boundary-layer growth over a flat plate with leading edge at $x = 0$ is given by

$$\delta \cong [5.83(\nu x/u_s)]^{1/2} \tag{A.4}$$

and the inlet length x_{in} is given by $(1/5.83)(u_s b/\nu)b$. The latter is less than a centimeter for the cases in flow models under consideration for a containment system. Therefore, the effect of the inlet length can be neglected for the flow range under consideration, and δ is taken as equal to b from the inlet. With $Q = (2/3)2\pi rbu_s$, Equation A.1 can be expressed as

$$u_s(du_s/dx) = (5/2)\{-2\nu(4\pi/3Q)^2 r^2 u_s^3 + g[1 + (dr/dz)^2]^{-1/2}\} \tag{A.5}$$

with x along the curved dome, with integration beginning from x_i, b_i and u_{si}, assuming parabolic velocity profile from the beginning. Alternately, Equation A.5 can be expressed as

$$u_s(du_s/dz) = (5/2)[-2\nu(4\pi/3Q)^2 r^2 u_s^3][1 + (dr/dz)^2]^{1/2} + (5/2)g \quad \text{(A.6)}$$

with the understanding that u_s is in the direction of x. Transition to the cylindrical section occurs as $dr/dz = 0$ and $dx = dz$. Note that the r^2 term in Equation A.6 gives $\sin^2\theta$ instead of $\sin\theta$ in Equation 24 where a conservative approximation of the friction factor based on terminal velocity over the cylinder of radius R was used. This effect is deemed minor when applied to thin films.

In the case of a 2-D vertical flat plate or cylinder of large radius as compared to the liquid film thickness, Equation A.6 for vertical coordinate z can be expressed in the form

$$(2/5)u_s(d/dz)u_s b = -2\nu(u_s/b) + bg \quad \text{(A.7)}$$

With the continuity condition $(2/3)u_s b = Q_L$ or $b = (3/2)Q_L/u_s$, it can be reduced to

$$u_s(du_s/dz) = -(20/9)(\nu/Q_L^2)u_s^3 + (5/2)g \quad \text{(A.8)}$$

Equation A.8 is integrable for constant θ by separation of variables. With the initial conditions $z = 0$, $u = u_{si}$, and $Q_L = (2/3)u_{si}b_i$, the terminal velocity $u_{st} = g\,b^2/2\nu$, and $u_{st} = [9gQ_L^2/\nu]^{1/3}/2 = (3/2)u_{mt}$, Equation A.8 can be expressed in the same form as Equation 28:

$$u^+\,du^+/dX = 1 - u^{+3} \quad \text{(A.9)}$$

Now, however, $X = (x/R)/R^*$ with $R^* = R_L^*$,

$$R_L^* = (9/10)[(Q/2\pi)^4/9R^7\nu^2 g]^{1/3} \quad \text{(A.10)}$$

This difference by a factor of 9/10 from Equation 22 accounts for the difference in the averaging procedure. For a turbulent liquid film, the difference of a corresponding factor from 1 is expected to be still smaller.

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